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NUMERICAL STUDY OF NAVIER–STOKES EQUATIONS

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The problem of a viscous incompressible fluid flow around a body of revolution at incidence, which is described by Navier–Stokes equations, is considered. For low Reynolds numbers, the solutions of these equations are smooth functions. A numerical algorithm without saturation is constructed, which responds to solution smoothness. The calculations are performed on grids consisting of 900 $(10 \times 10 \times 9)$ *and* 700 $(10 \times 10 \times 7)$ *nodes. On the grid consisting of* 900 *nodes, a system of* 3600 *nonlinear equations is solved by a standard code. The pressures on the shaded side of the body of revolution are compared. It is found that a numerical study (on this grid) is feasible for problems with* $\mathrm{Re} \approx 1$. *For high Reynolds numbers, the number of grid nodes has to be increased.*

Key words: *Navier–Stokes equations, viscous fluid flow, numerical algorithm without saturation.*

Introduction. Stokes equations were considered earlier in [1]. In the present paper, those results are extended to Navier–Stokes equations. Full nonlinear Navier–Stokes equations (for an incompressible viscous fluid) in the exterior of a body of revolution are considered for the case with an arbitrary direction of the flow velocity vector with respect to the axis of revolution:

$$\mu\Delta u - \rho u \cdot \nabla u - \nabla p = 0, \qquad x \in \mathbb{R}^3 \setminus \Omega,$$
$$\nabla \cdot u = 0. \tag{1}$$

Here $u(x)$ is the velocity of the incompressible fluid, $p(x)$ is the pressure, μ and ρ are the fluid viscosity and density, and Ω is the domain under consideration. Equations (1) are defined in the exterior of the domain Ω.

The system of differential equations (1) is supplemented by the boundary conditions

$$|x| \to \infty: \qquad u \to u_\infty,$$
$$u\Big|_{\partial\Omega} = 0; \tag{2}$$

$$|x| \to \infty: \qquad p \to p_\infty \tag{3}$$

($\partial\Omega$ is the boundary of the domain Ω).

The problem usually considered in the literature is problem (1), (2). In this case, the solution is not unique (the value of p is determined with accuracy to a constant). Therefore, problem (1)–(3) is considered below, i.e., the pressure at infinity is assumed to be constant and independent of x.

Using some physically feasible assumptions, Leray established the existence of rigorous solutions of the interior problem and the exterior problem with $u_\infty = 0$ [2]. In the case of the exterior problem with $u_\infty \neq 0$, the solution satisfies (1), (2); instead of the first relation in (2), however, weaker conditions

$$\int\limits_R |\nabla u|^2 \, dx < \infty, \qquad \int\limits_R \frac{|u(y) - u_\infty|^2}{|x - y|^2} \, dy \leqslant K < \infty$$

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with a constant K independent of x are satisfied. Finn [3] proved that Leray's solution also satisfies the condition at infinity in the case $u_\infty \neq 0$.

Other possible proofs were proposed in [4, 5]. Note that the system of Navier–Stokes equations is elliptical by definition given by Douglis and Nirenberg [6]; hence, u and p are known to be smooth to a certain extent.

There are many publications dealing with numerical solutions of Navier–Stokes equations. The closest paper to the present one is that published by Babenko et al. [7] who considered a two-dimensional problem on a viscous incompressible fluid flow around a cylinder. The boundary conditions at infinity were shifted to a large-diameter circumference, and a time-dependent method was used to solve nonlinear equations resulting from discretization. Note that this method does not provide convergence to the solution of the exterior problem as the radius of the outer circumference R increases. In the exterior of the circle, the Laplace operator entering the Navier–Stokes equations has a continuous spectrum filling the interval $(-\infty, 0)$. The calculations produce a discrete spectrum, but the eigenvalues of the corresponding matrix (matrix of the discrete Laplace operator) have two condensation points: 0 and $-\infty$. Thus, the norm of the matrix of the discrete Laplace operator has a high value (as well as the norm of the inverse matrix). As a result, the time-dependent method is unstable in the limit as $R \to \infty$.

The method most frequently used to solve Navier–Stokes equations is the finite-element method (see, e.g., [8] and the references therein). The greatest difficulty in using the finite-element method is the necessity of solving a large number of nonlinear equations. An alternative to the finite-element method, namely, discretization without saturation [9], is used in the present work. The point is that the solutions of Navier–Stokes equations for low Reynolds numbers are smooth functions, and one has to use this circumstance, i.e., to construct an algorithm responding to solution smoothness [9]. As a result, a three-dimensional problem on the flow around a body of revolution at incidence can be numerically studied on a grid consisting of $900 = 10 \times 10 \times 9$ nodes. The resultant system of 3600 nonlinear equations is solved by applying a standard code [10].

1. Formulation of the Problem. Full steady-state Navier–Stokes equations are considered in the exterior of a body of revolution Ω. These equations have the form [11]

$$
\begin{gathered}
u^1 \frac{\partial u^1}{\partial x_1} + u^2 \frac{\partial u^1}{\partial x_2} + u^3 \frac{\partial u^1}{\partial x_3} = -\frac{\partial p}{\partial x_1} + \frac{1}{\mathrm{Re}} \nabla^2 u^1, \\
u^1 \frac{\partial u^2}{\partial x_1} + u^2 \frac{\partial u^2}{\partial x_2} + u^3 \frac{\partial u^2}{\partial x_3} = -\frac{\partial p}{\partial x_2} + \frac{1}{\mathrm{Re}} \nabla^2 u^2, \\
u^1 \frac{\partial u^3}{\partial x_1} + u^2 \frac{\partial u^3}{\partial x_2} + u^3 \frac{\partial u^3}{\partial x_3} = -\frac{\partial p}{\partial x_3} + \frac{1}{\mathrm{Re}} \nabla^2 u^3, \\
\frac{\partial u^1}{\partial x_1} + \frac{\partial u^2}{\partial x_2} + \frac{\partial u^3}{\partial x_3} = 0.
\end{gathered}
\tag{1.1}
$$

Here (u^1, u^2, u^3) is the velocity vector, u^i $(i = 1, 2, 3)$ are the projections of the velocity vector onto the axes of the Cartesian coordinate system (x_1, x_2, x_3), and ∇^2 is the Laplace operator.

System (1.1) has to be supplemented by the boundary conditions for u^i $(i = 1, 2, 3)$ and the pressure p.

In system (1.1), we use the following dimensionless quantities (the bar is omitted): $x_i = \bar{x}_i/L_a$, $u^i = \bar{u}^i/u_\infty$, $p = (\bar{p} - p_\infty)/(\rho v_\infty^2)$, and $1/\mathrm{Re} = \nu/(L_a u_\infty)$ (L_a is the characteristic linear size, u_∞ is the absolute value of flow velocity at infinity, p_∞ is the pressure at infinity, ν is the viscosity, and Re is the Reynolds number).

Thus, to determine the flow parameters, velocity vector (u^1, u^2, u^3), and pressure p, we need to solve system (1.1) subjected to the following boundary conditions:

$$
u^i\Big|_{\partial\Omega} = 0, \quad u^i\Big|_{\infty} = u^i_\infty, \quad i = 1, 2, 3, \qquad p\Big|_{\infty} = 0.
$$

Here Ω is the domain occupied by the body under consideration, which is formed by means of rotation around the x_3 axis, $\partial\Omega$ is the boundary of the domain Ω, and u^i_∞ $(i = 1, 2, 3)$ is the fluid velocity in an undisturbed flow (at infinity).

We introduce a system of curvilinear coordinates (r, θ, φ) related to the Cartesian coordinates (x_1, x_2, x_3) by

$$
x_1 = V(r, \theta) \cos\varphi, \qquad x_2 = V(r, \theta) \sin\varphi, \qquad x_3 = U(r, \theta). \tag{1.2}
$$

The domain obtained in the meridional cross section of the body Ω is denoted by G. The functions U and V are chosen in the following manner. Let $\psi = \psi(z)$, $\psi = U + iV$, $z = r\exp(i\theta)$ be a conformal mapping of the circle $|z| \leqslant 1$ onto the exterior of the domain G, the center of the circle being mapped as an infinitely distant point (for the conformal mapping to be unique, we require that the direction along the real axis at the zero point transform to the same direction). Relations (1.2) define the image of a sphere of a unit radius onto the exterior of the body Ω.

For an ellipsoid of revolution around the x_3 axis ($x_1^2/b^2 + x_2^2/b^2 + x_3^2/a^2 - 1 = 0$), the functions U and V are known in analytical form (see [12]). After mapping (1.2), the surface of a sphere of a unit radius transforms to the body surface Ω. The boundary conditions specified on $\partial\Omega$ are thereby shifted to the sphere surface, while the boundary conditions prescribed at infinity are shifted to the sphere center. Note that such mapping converts the outward normal to the body into the inward normal. This fact is important to calculate the drag force (see below).

If curvilinear coordinates are used, the equations for vector quantities are normally written in projections onto the axes of the own basis whose coordinate vectors are directed along the tangents to the coordinate lines. This basis depends on the coordinates of the point in space. In our case, this approach is unreasonable, because mapping (1.2) on the x_3 axis is not unique (if $V = 0$, then φ can take arbitrary values), which is responsible for solution singularities caused by using a "bad" coordinate system. Note that a spherical coordinate system has the same drawback.

This situation can be resolved as follows. We retain the projections of the velocity vector u^i ($i = 1, 2, 3$) onto the axes of the Cartesian coordinate system as the sought functions and replace the independent variables x_1, x_2, and x_3 by r, θ, and φ by means of substitution (1.2). Then, the partial derivatives with respect to the Cartesian coordinates x_i ($i = 1, 2, 3$) are expressed via the derivatives with respect to r, θ, and φ [$\Phi(x_1, x_2, x_3) = \Phi(V\cos\varphi, V\sin\varphi, U)$]:

$$\frac{\partial\Phi}{\partial x_1} = \alpha\cos\varphi\,\frac{\partial\Phi}{\partial r} + \beta\cos\varphi\,\frac{\partial\Phi}{\partial\theta} - \frac{1}{V}\,\sin\varphi\,\frac{\partial\Phi}{\partial\varphi},$$

$$\frac{\partial\Phi}{\partial x_2} = \alpha\sin\varphi\,\frac{\partial\Phi}{\partial r} + \beta\sin\varphi\,\frac{\partial\Phi}{\partial\theta} + \frac{1}{V}\,\cos\varphi\,\frac{\partial\Phi}{\partial\varphi}, \tag{1.3}$$

$$\frac{\partial\Phi}{\partial x_3} = \frac{rV'_\theta}{w^2}\,\frac{\partial\Phi}{\partial r} - \frac{rV'_r}{w^2}\,\frac{\partial\Phi}{\partial\theta}.$$

Here $\alpha(r,\theta) = -rU'_\theta/w^2$, $\beta(r,\theta) = (1 + rU'_\theta V'_r/w^2)/V'_\theta$, and $w^2 = U'^2_\theta + V'^2_\theta$. As a result, we obtain

$$\Big(u^1\alpha\cos\varphi + u^2\alpha\sin\varphi + u^3\,\frac{rV'_\theta}{w^2}\Big)\frac{\partial u^1}{\partial r} + \Big(u^1\beta\cos\varphi + u^2\beta\sin\varphi - u^3\,\frac{rV'_r}{w^2}\Big)\frac{\partial u^1}{\partial\theta}$$

$$+ \Big(-\frac{1}{V}\,\sin\varphi\,u^1 + \frac{1}{V}\,\cos\varphi\,u^2\Big)\frac{\partial u^1}{\partial\varphi} + \Big(\alpha\cos\varphi\,\frac{\partial p}{\partial r} + \beta\cos\varphi\,\frac{\partial p}{\partial\theta} - \frac{1}{V}\,\sin\varphi\,\frac{\partial p}{\partial\varphi}\Big) = \frac{1}{\mathrm{Re}}\,\nabla^2 u^1,$$

$$\Big(u^1\alpha\cos\varphi + u^2\alpha\sin\varphi + u^3\,\frac{rV'_\theta}{w^2}\Big)\frac{\partial u^2}{\partial r} + \Big(u^1\beta\cos\varphi + u^2\beta\sin\varphi - u^3\,\frac{rV'_r}{w^2}\Big)\frac{\partial u^2}{\partial\theta}$$

$$+ \Big(-\frac{1}{V}\,\sin\varphi u^1 + \frac{1}{V}\,\cos\varphi u^2\Big)\frac{\partial u^3}{\partial\varphi} + \Big(\alpha\sin\varphi\,\frac{\partial p}{\partial r} + \beta\sin\varphi\,\frac{\partial p}{\partial\theta} + \frac{1}{V}\,\cos\varphi\,\frac{\partial p}{\partial\varphi}\Big) = \frac{1}{\mathrm{Re}}\,\nabla^2 u^2,$$

$$\Big(u^1\alpha\cos\varphi + u^2\alpha\sin\varphi + u^3\,\frac{rV'_\theta}{w^2}\Big)\frac{\partial u^3}{\partial r} + \Big(u^1\beta\cos\varphi + u^2\beta\sin\varphi - u^3\,\frac{rV'_r}{w^2}\Big)\frac{\partial u^3}{\partial\theta} \tag{1.4}$$

$$+ \Big(-\frac{1}{V}\,\sin\varphi u^1 + \frac{1}{V}\,\cos\varphi u^2\Big)\frac{\partial u^3}{\partial\varphi} + \Big(\frac{rV'_\theta}{w^2}\,\frac{\partial p}{\partial r} - \frac{rV'_r}{w^2}\,\frac{\partial p}{\partial\theta}\Big) = \frac{1}{\mathrm{Re}}\,\nabla^2 u^3,$$

$$\alpha\cos\varphi\,\frac{\partial u^1}{\partial r} + \beta\cos\varphi\,\frac{\partial u^1}{\partial\theta} - \frac{1}{V}\,\sin\varphi\,\frac{\partial u^1}{\partial\varphi} + \alpha\sin\varphi\,\frac{\partial u^2}{\partial r} + \beta\sin\varphi\,\frac{\partial u^2}{\partial\theta} + \frac{1}{V}\,\cos\varphi\,\frac{\partial u^2}{\partial\varphi} + \frac{rV'_\theta}{w^2}\,\frac{\partial u^3}{\partial r} - \frac{rV'_r}{w^2}\,\frac{\partial u^3}{\partial\theta} = 0.$$

Relations (1.4) can be brought to uniform equations for velocity by the replacement

$$u^i = (1-r)u^i_\infty + \hat{u}^i, \qquad \hat{u}^i\Big|_{r=0} = \hat{u}^i\Big|_{r=1} = 0,$$
$$\frac{\partial u^i}{\partial r} = -u^i_\infty + \frac{\partial \hat{u}^i}{\partial r}, \qquad \Delta u^i = \Delta\hat{u}^i - \frac{r}{w^2}\Big(1 + r\,\frac{V'_r}{V}\Big)u^i_\infty, \qquad i = 1,2,3. \tag{1.5}$$

The sought functions u^i are replaced by $\hat{u}^i$ $(i = 1,2,3)$ with the use of Eq. (1.5) to obtain uniform boundary conditions for velocity:

$$\hat{u}^i\Big|_{r=0} = \hat{u}^i\Big|_{r=1} = 0, \qquad i = 1,2,3. \tag{1.6}$$

This is necessary for a more convenient discretization of the Laplacian. For the pressure, we have the boundary condition

$$p\Big|_{r=0} = 0. \tag{1.7}$$

The Laplacian of the functions u^i $(i = 1,2,3)$ in the variables r, θ, φ acquires the form

$$\Delta u^i = \frac{r}{Vw^2}\Big[\frac{\partial}{\partial r}\Big(rV\,\frac{\partial u^i}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{V}{r}\,\frac{\partial u^i}{\partial\theta}\Big)\Big] + \frac{1}{V^2}\,\frac{\partial^2 u^i}{\partial\varphi^2}. \tag{1.8}$$

Thus, we have to solve Eqs. (1.4) and (1.5) in a sphere of a unit radius with the boundary conditions (1.6) and (1.7).

2. Discrete Laplacian. For discretization of Laplacian (1.8) with uniform boundary conditions (1.6), we use a technique described in [1]. The essence of this technique is as follows. In Laplacian (1.8), using the method of separation of variables, one can reduce calculating the eigenvalues of a three-dimensional differential operator to calculating the eigenvalues of two-dimensional differential operators. The discrete problem inherits these properties, and discretization of a three-dimensional differential operator reduces to discretization of a series of two-dimensional operators.

Thus, we obtain a discrete Laplacian in the form of an h-matrix:

$$H = \frac{2}{L}\sum_{k=0}^{l}{}' \Lambda_k \otimes h_k, \qquad L = 2l+1.$$

The prime here means that the term at $k = 0$ is taken with a coefficient equal to 1/2; the sign "$\otimes$" denotes the Kronecker product of matrices; h is a matrix of $L \times L$ size with the elements

$$h_{kij} = \cos k\,\frac{2\pi(i-j)}{L} \qquad (i,j = 1,2,\ldots,L);$$

Λ_k is the matrix of the discrete operator corresponding to the differential operator

$$\frac{r}{Vw^2}\Big[\frac{\partial}{\partial r}\Big(rV\,\frac{\partial\Phi}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{V}{r}\,\frac{\partial\Phi}{\partial\theta}\Big)\Big] - \frac{k^2}{V^2}\,\Phi, \qquad k = 0,1,\ldots,l \tag{2.1}$$

with the boundary conditions

$$\Phi\Big|_{r=0} = \Phi\Big|_{r=1} = 0. \tag{2.2}$$

For discretization of the differential operator (2.1), (2.2), we choose a grid consisting of n nodes in terms of θ

$$\theta_s = \frac{\pi}{2}\,(y_s+1) \qquad \Big(y_s = \cos\varepsilon_s, \qquad \varepsilon_s = \frac{(2s-1)\pi}{2n}, \qquad s = 1,2,\ldots,n\Big)$$

and use the interpolation formula

$$g(\theta) = \sum_{s=1}^{n}\frac{T_n(y)g_s}{n(-1)^{s-1}(y-y_s)/\sin\varepsilon_s}$$
$$\Big[= \frac{1}{\pi}\,(2\theta-\pi), g_s = g(\theta_s), \quad s = 1,2,\ldots,n, \qquad T_n(y) = \cos\,(n\arccos y)\Big]. \tag{2.3}$$

The first and second derivatives with respect to θ, which enter relations (2.1), are obtained by differentiating the interpolation formula (2.3).

In terms of r, we choose a grid consisting of m nodes

$$r_s = \frac{1}{2}(z_s + 1) \qquad \Bigl(z_s = \cos\chi_s, \qquad \chi_s = \frac{(2s-1)\pi}{2m}, \qquad s = 1, 2, \dots, m\Bigr)$$

and use the interpolation formula

$$q(r) = \sum_{s=1}^{m} \frac{T_m(r)(r-1)rq_s}{m(-1)^{s-1}(r_s-1)r_s(z-z_s)/\sin\chi_s} \qquad \Bigl[q_s = q(r_s), \qquad z = 2r-1\Bigr]. \tag{2.4}$$

The first and second derivatives with respect to r, which enter Eq. (2.1), are found by differentiating the interpolation formula (2.4). By differentiating the interpolation formulas (2.3) and (2.4), we obtain the values of the derivatives with respect to θ and r, which enter the left side of the continuity equation (1.8).

For discretization of the derivatives of pressure with respect to r, we use the interpolation formula

$$q(r) = \sum_{s=1}^{m} \frac{T_m(r)rq_s}{m(-1)^{s-1}r_s(z-z_s)/\sin\chi_s}, \qquad z = 2r-1. \tag{2.5}$$

The quantities entering Eq. (2.5) are defined above. The values of the first derivative of pressure with respect to r, which enter the left side of relations (1.5)–(1.7), are obtained by differentiating the interpolation formula (2.5).

To construct a formula for numerical differentiation with respect to φ, we consider the interpolation formula

$$\begin{gathered} S(\varphi) = \frac{2}{L}\sum_{k=0}^{2l} D_l(\varphi-\varphi_k)S_k, \qquad L = 2l+1, \qquad S_k = S(\varphi_k), \\ \varphi_k = \frac{2\pi k}{L} \quad (k = 0, 1, \dots, 2l), \qquad D_l(\varphi-\varphi_k) = \frac{1}{2} + \sum_{j=1}^{l}\cos j(\varphi-\varphi_k). \end{gathered} \tag{2.6}$$

The values of the derivatives with respect to φ are determined by differentiating formula (2.6).

To derive discrete Navier–Stokes equations, we need to replace the derivatives in Eqs. (1.4) by discrete derivatives found by differentiating the corresponding interpolation formulas (2.3)–(2.6); the Laplacian is replaced by the matrix H. Instead of the functions u^1, u^2, u^3, and p, discrete Stokes equations contain the values of these functions in grid nodes $(\theta_\nu, r_\mu, \varphi_k)$, $\nu = 1, 2, \dots, n$, $\mu = 1, 2, \dots, m$, and $k = 0, 1, \dots, 2l$.

As a result, we obtain a system of $4mnL$ nonlinear equations. The system of discrete equations in explicit form is written below. For $m = n = 10$ and $L = 9$, for instance, the system contains 3600 equations.

Let us comment on the Laplacian discretization used. Interpolation of the solution by polynomials is applied. It is known that such an interpolation responds to solution smoothness, and the greater the number of smoothness conditions satisfied by the interpolation, the better the approximation of the interpolated function [9]. In this case, one does not need to known the solution smoothness *a priori*. This is an essential feature of numerical algorithms without saturation [9]. As was noted above, the solution in the problem under study consists of smooth functions. Using this circumstance, one can construct an algorithm with acceptable accuracy on a sparse grid.

3. Discrete Navier–Stokes Equations. The following notation is introduced:

$$\begin{gathered} A^{(1)} = A^{(1)}(u^1, u^2, u^3) = u^1\alpha\cos\varphi + u^2\alpha\sin\varphi + u^3 rV'_\theta/w^2, \\ A^{(2)} = A^{(2)}(u^1, u^2, u^3) = u^1\beta\cos\varphi + u^2\beta\sin\varphi - u^3 rV'_r/w^2, \\ A^{(3)} = A^{(3)}(u^1, u^2, u^3) = -(1/V)\sin\varphi\, u^1 + (1/V)\cos\varphi\, u^2. \end{gathered}$$

The functions α and w^2 are defined in (1.3). We use substitution (1.5) and introduce new notations:

$$\begin{aligned} \hat{A}^{(1)} = \hat{A}^{(1)}(u^1, u^2, u^3) &= (1-r)u^1_\infty\alpha\cos\varphi + (1-r)u^2_\infty\alpha\sin\varphi + (1-r)u^3_\infty rV'_\theta/w^2 \\ &\quad + \hat{u}^1\alpha\cos\varphi + \hat{u}^2\alpha\sin\varphi + \hat{u}^3 rV'_\theta/w^2, \\ \hat{A}^{(2)} = \hat{A}^{(2)}(u^1, u^2, u^3) &= (1-r)u^1_\infty\beta\cos\varphi + (1-r)u^2_\infty\beta\sin\varphi - (1-r)u^3_\infty rV'_r/w^2 \\ &\quad + \hat{u}^1\beta\cos\varphi + \hat{u}^2\beta\sin\varphi - \hat{u}^3 rV'_r/w^2, \end{aligned}$$

$$\hat{A}^{(3)} = \hat{A}^{(3)}(u^1, u^2, u^3) = -(1/V)\sin\varphi(1-r)u^1_\infty + (1/V)\cos\varphi(1-r)u^2_\infty - (1/V)\sin\varphi\,\hat{u}^1 + (1/V)\cos\varphi\,\hat{u}^2.$$

We use θ_ν, r_μ, φ_k ($\nu = 1, 2, \ldots, n$; $\mu = 1, 2, \ldots, m$; $k = 0, 1, \ldots, 2l$, $L = 2l + 1$) to denote the values of the variables θ, r, and φ in grid nodes. Then, Eqs. (1.4) yield discrete Navier–Stokes equations:

$$\begin{aligned}
&\hat{A}^{(1)}_{\nu\mu k}\Big(-u^1_\infty + \sum_{\mu_1=1}^{m} D^{(r)}_{\mu\mu_1}\hat{u}^1_{\nu\mu_1 k}\Big) + \hat{A}^{(2)}_{\nu\mu k}\Big(\sum_{\nu_1=1}^{n} D^{(\theta)}_{\nu\nu_1}\hat{u}^1_{\nu_1\mu k}\Big) + \hat{A}^{(3)}_{\nu\mu k}\Big(\sum_{k_1=0}^{2l} D^{(\varphi)}_{kk_1}\hat{u}^1_{\nu\mu k_1}\Big)\\
&+ \alpha_{\mu\nu}\cos\varphi_k\Big(\sum_{\mu_1=1}^{m} D^{(pr)}_{\mu\mu_1}p_{\nu\mu_1 k}\Big) + \beta_{\mu\nu}\cos\varphi_k\Big(\sum_{\nu_1=1}^{n} D^{(\theta)}_{\nu\nu_1}p_{\nu_1\mu k}\Big) - \frac{1}{V_{\mu\nu}}\sin\varphi_k\Big(\sum_{k_1=0}^{2l} D^{(\varphi)}_{kk_1}p_{\nu\mu k_1}\Big)\\
&- \frac{1}{\mathrm{Re}}\Big(\sum_{\nu_1=1}^{n}\sum_{\mu_1=1}^{m}\sum_{k_1=0}^{2l} H_{\nu\mu k,\nu_1\mu_1 k_1}\hat{u}^1_{\nu_1\mu_1 k_1} - \frac{r}{w^2}\Big(1 + r\frac{V'_r}{V}\Big)\Big|_{\substack{r=r_\mu\\ \theta=\theta_\nu}} u^1_\infty\Big),\\
&\hat{A}^{(1)}_{\nu\mu k}\Big(-u^2_\infty + \sum_{\mu_1=1}^{m} D^{(r)}_{\mu\mu_1}\hat{u}^2_{\nu\mu_1 k}\Big) + \hat{A}^{(2)}_{\nu\mu k}\Big(\sum_{\nu_1=1}^{n} D^{(\theta)}_{\nu\nu_1}\hat{u}^2_{\nu_1\mu k}\Big) + \hat{A}^{(3)}_{\nu\mu k}\Big(\sum_{k_1=0}^{2l} D^{(\varphi)}_{kk_1}\hat{u}^2_{\nu\mu k_1}\Big)\\
&+ \alpha_{\mu\nu}\sin\varphi_k\Big(\sum_{\mu_1=1}^{m} D^{(pr)}_{\mu\mu_1}p_{\nu\mu_1 k}\Big) + \beta_{\mu\nu}\sin\varphi_k\Big(\sum_{\nu_1=1}^{n} D^{(\theta)}_{\nu\nu_1}p_{\nu_1\mu k}\Big) + \frac{1}{V_{\mu\nu}}\cos\varphi_k\Big(\sum_{k_1=0}^{2l} D^{(\varphi)}_{kk_1}p_{\nu\mu k_1}\Big)\\
&- \frac{1}{\mathrm{Re}}\Big(\sum_{\nu_1=1}^{n}\sum_{\mu_1=1}^{m}\sum_{k_1=0}^{2l} H_{\nu\mu k,\nu_1\mu_1 k_1}\hat{u}^2_{\nu_1\mu_1 k_1} - \frac{r}{w^2}\Big(1 + r\frac{V'_r}{V}\Big)\Big|_{\substack{r=r_\mu\\ \theta=\theta_\nu}} u^2_\infty\Big), \qquad (3.1)\\
&\alpha_{\mu\nu}\cos\varphi_k\Big(\sum_{\mu_1=1}^{m} D^{(r)}_{\mu\mu_1}\hat{u}^1_{\nu\mu_1 k}\Big) + \beta_{\mu\nu}\cos\varphi_k\Big(\sum_{\nu_1=1}^{n} D^{(\theta)}_{\nu\nu_1}\hat{u}^1_{\nu_1\mu k}\Big) - \frac{1}{V_{\mu\nu}}\sin\varphi_k\Big(\sum_{k_1=0}^{2l} D^{(\varphi)}_{kk_1}\hat{u}^1_{\nu\mu k_1}\Big)\\
&+ \alpha_{\mu\nu}\sin\varphi_k\Big(\sum_{\mu_1=1}^{m} D^{(r)}_{\mu\mu_1}\hat{u}^2_{\nu\mu_1 k}\Big) + \beta_{\mu\nu}\sin\varphi_k\Big(\sum_{\nu_1=1}^{n} D^{(\theta)}_{\nu\nu_1}\hat{u}^2_{\nu_1\mu k}\Big) + \frac{1}{V_{\mu\nu}}\cos\varphi_k\Big(\sum_{k_1=0}^{2l} D^{(\varphi)}_{kk_1}\hat{u}^2_{\nu\mu k_1}\Big)\\
&+ \frac{rV'_\theta}{w^2}\Big|_{\substack{r=r_\mu\\ \theta=\theta_\nu}}\Big(\sum_{\mu_1=1}^{m} D^{(pr)}_{\mu\mu_1}\hat{u}^3_{\nu\mu_1 k}\Big) - \frac{rV'_r}{w^2}\Big|_{\substack{r=r_\mu\\ \theta=\theta_\nu}}\Big(\sum_{\nu_1=1}^{n} D^{(\theta)}_{\nu\nu_1}\hat{u}^3_{\nu_1\mu k}\Big)\\
&- \Big(u^1_\infty\alpha_{\mu\nu}\cos\varphi_k + u^2_\infty\alpha_{\mu\nu}\sin\varphi_k + u^3_\infty\frac{rV'_\theta}{w^2}\Big|_{\substack{r=r_\mu\\ \theta=\theta_\nu}}\Big) = 0.
\end{aligned}$$

From Eqs. (3.1), we need to determine the vector $(\hat{u}^1, \hat{u}^2, \hat{u}^3, p)'$, where $\hat{u}^i$ and p are the vectors of the values of the corresponding functions in grid nodes.

Remark 1. $D^{(r)}$, $D^{(pr)}$, $D^{(\theta)}$, and $D^{(\varphi)}$ are the matrices of numerical differentiation, which are obtained by differentiating the interpolation formulas (2.4), (2.5), (2.3), and (2.6), respectively. The programs for calculating these matrices are described in [13].

4. Calculation Results. The calculations were performed for a sphere of a unit radius and a velocity vector at infinity equal to $u = (1, 0, 0)$, and also for an ellipsoid ($a = 1$, $b = 0.5$) with two directions of the velocity vector at infinity $u = (1, 0, 0)$ and $u = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ on grids consisting of 900 ($10 \times 10 \times 9$) and 700 ($10 \times 10 \times 7$) nodes. The pressure on the shaded side of the sphere (ellipsoid), which was calculated on a grid consisting of 900 nodes for different values of θ and $\varphi = 0$, are summarized in Table 1 (Δ is the relative error, as compared with the calculations on a grid consisting of 700 nodes). The following values were used in these problems: $u = (1, 0, 0)$, $\mathrm{Re} = 0.1$, $n = 10$, $m = 10$, and $L = 8$.

TABLE 1

Pressure on the Shaded Side of a Sphere and an Ellipsoid

Node number	Sphere		Ellipsoid	
	p	Δ, %	p	Δ, %
1	−0.4649	74.10	0.6820	90.31
2	2.0966	0.88	12.3750	0.04
3	6.8262	0.27	24.3430	0.006
4	12.6380	0.41	25.1352	1.10
5	16.9072	1.50	23.4242	2.30
6	16.9072	1.50	23.4242	2.30
7	12.6380	0.41	25.1352	1.10
8	6.8262	0.27	24.3430	0.006
9	2.0966	0.88	12.3750	0.04
10	−0.4649	74.10	0.6820	90.31

The calculations were performed until the value $\mathrm{Re} = 1$ was reached. With increasing Reynolds number, the calculation accuracy decreases and becomes unacceptable at $\mathrm{Re} > 1$. The number of grid nodes cannot be increased because of the failure in operation of the standard code for solving nonlinear equations.

The results in Table 1 imply that the solution reaches the highest error at points close to sphere poles. Then the accuracy increases but again decreases with approaching the x axis.

5. Drag Calculations. The drag coefficient c_x was calculated for a sphere of a unit radius [11]. The projection of the force acting on the sphere onto the x axis is

$$F_x = \int\limits_{\Sigma} (p_{11}n_1 + p_{12}n_2 + p_{13}n_3)\, d\sigma.$$

Here p_{ij} are the components of the stress tensor:

$$p_{11} = -p + 2\mu \frac{\partial u^1}{\partial x_1}, \qquad p_{12} = \mu\Big(\frac{\partial u^1}{\partial x_2} + \frac{\partial u^2}{\partial x_1}\Big), \qquad p_{13} = \mu\Big(\frac{\partial u^1}{\partial x_3} + \frac{\partial u^3}{\partial x_1}\Big);$$

in the case of a sphere, $n_1 = \sin\theta\cos\varphi$, $n_2 = \sin\theta\sin\varphi$, and $n_3 = \cos\theta$.

The partial derivatives with respect to x_i are expressed in terms of the derivatives with respect to r, θ, and φ with the use of formula (1.3). Taking into account that the derivatives with respect to θ and φ equal zero on the sphere surface, we obtain the following relations by virtue of the boundary conditions:

$$p_{1j}n_j = -p\sin\theta\cos\varphi + \mu\Big[(\sin^2\theta\cos^2\varphi - \cos 2\theta)\,\frac{\partial u^1}{\partial r} + \sin^2\theta\sin\varphi\cos\varphi\,\frac{\partial u^2}{\partial r} + \sin\theta\cos\theta\cos\varphi\,\frac{\partial u^3}{\partial r}\Big],$$

$$d\sigma = R^2\sin\theta\, d\theta\, d\varphi, \qquad F_x = \int\limits_0^{\pi} \Big(\sin\theta \int\limits_0^{2\pi} p_{1j}n_j\, d\varphi\Big)\, d\theta = \frac{2\pi}{L}\sum_{s=1}^{n} c_s \sum_{k=0}^{2l} f_{sk}.$$

Here c_s are the coefficients of the quadrature formula with respect to θ on the interval $[0, \pi]$:

$$c_s = \frac{\pi}{n}\Big(1 - 2\sum_{l=2(2)}^{n-1} \frac{\cos l\psi_s}{l^2 - 1}\Big), \qquad \psi_s = \frac{(2s-1)\pi}{2n}, \qquad s = 1, 2, \dots, n,$$

$$f_{sk} = f(\theta_s, \varphi_k), \qquad \theta_s = \frac{(2s-1)\pi}{2m}, \quad s = 1, 2, \dots, m, \qquad \varphi_k = \frac{2\pi k}{L}, \quad k = 0, \dots, 2l,$$

$$f = -p\sin^2\theta\cos\varphi + \frac{1}{\mathrm{Re}}\Big[(\sin^3\theta\cos^2\varphi - \cos 2\theta\sin\theta)\Big(-1 + \frac{\partial\hat u^1}{\partial r}\Big) + \sin^3\theta\sin\varphi\cos\varphi\,\frac{\partial\hat u^2}{\partial r} + \sin^2\theta\cos\theta\cos\varphi\,\frac{\partial\hat u^3}{\partial r}\Big];$$

$l = 2(2)$ indicate summation from $l = 2$ with a step equal to two.

Remark 2. The calculations by the above-given formula yield a negative drag coefficient, because the outward normal to the sphere is transformed by mapping (1.2) into the inward normal. This fact is taken into account in the code (the sign of f is changed).

We have still to give formulas for calculating p and the derivatives of the velocity components on the sphere surface:

$$p(1) = \frac{4}{m} \sum_{s=1}^{m} \frac{c_s p_s}{1 + x_s}, \qquad u'(1) = \frac{8}{m} \sum_{s=1}^{m} \frac{c_s u_s}{x_s^2 - 1},$$

$$c_s = \frac{1}{2} + \sum_{l=1}^{m-1} \cos l\theta_s, \qquad x_s = \cos\theta_s, \qquad \theta_s = \frac{(2s-1)\pi}{2m}, \qquad s = 1, 2, \ldots, m.$$

Here p_s and u_s are the values of pressure and velocity at grid nodes with respect to r.

We calculated the values of

$$c_x = \frac{F_x}{\rho u_\infty^2 \pi R^2/2} = \frac{4}{L} \sum_{s=1}^{n} c_s \sum_{k=0}^{2l} f_{sk}$$

on a grid consisting of 900 nodes. The values $c_x = 277.55$, 63.73, 23.99, and 13.60 were obtained for Re = 0.02655, 0.12185, 0.36385, and 0.74650, respectively.

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